

# CHARACTERISTIC NUMBER ASSOCIATED TO MASS LINEAR PAIRS

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ABSTRACT. Let  $\Delta$  be a Delzant polytope in  $\mathbb{R}^n$  and  $\mathbf{b} \in \mathbb{Z}^n$ . Let  $E$  denote the symplectic fibration over  $S^2$  determined by the pair  $(\Delta, \mathbf{b})$ . Under certain hypotheses, we prove the equivalence between the fact that  $(\Delta, \mathbf{b})$  is a mass linear pair (D. McDuff, S. Tolman, *Polytopes with mass linear functions. I.* Int. Math. Res. Not. IMRN 8 (2010) 1506-1574.) and the vanishing of a characteristic number of  $E$ . Denoting by  $\text{Ham}(M_\Delta)$  the Hamiltonian group of the symplectic manifold defined by  $\Delta$ , we determine loops in  $\text{Ham}(M_\Delta)$  that define infinite cyclic subgroups in  $\pi_1(\text{Ham}(M_\Delta))$ , when  $\Delta$  satisfies any of the following conditions: (i) it is the trapezium associated with a Hirzebruch surface, (ii) it is a  $\Delta_p$  bundle over  $\Delta_1$ , (iii)  $\Delta$  is the truncated simplex associated with the one point blow up of  $\mathbb{C}P^n$ .

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## 1. INTRODUCTION

Let  $(N, \Omega)$  be a closed connected symplectic  $2n$ -manifold. By  $\text{Ham}(N, \Omega)$ , we denote the Hamiltonian group of  $(N, \Omega)$  [6, 8]. Associated with a loop  $\psi$  in  $\text{Ham}(N, \Omega)$ , there exist characteristic numbers which are invariant under deformation of  $\psi$ . These invariants are defined in terms of characteristic classes of fibre bundles and their explicit values are not easy to calculate, in general. Here, we will consider a particular invariant  $I$ , whose definition we will recall below. By proving the non-vanishing of  $I$  for certain loops, we will deduce the existence of infinity cyclic subgroups of  $\pi_1(\text{Ham}(N, \Omega))$ , when  $N$  is a toric manifold. The vanishing of the invariant  $I$  on particular loops in  $\text{Ham}(N, \Omega)$  is related with the concept of mass linear pair, which has been developed in [7]. In this introduction, we will state the main results of the paper and will give a schematic exposition of the concepts involved in these statements.

A loop  $\psi$  in  $\text{Ham}(N, \Omega)$  determines a Hamiltonian fibre bundle  $E \rightarrow S^2$  with standard fibre  $N$ , via the clutching construction. Various characteristic numbers for the fibre bundle  $E$  have been defined in [4]. These numbers give rise to topological invariants of the loop  $\psi$ . In this article, we will consider only the following characteristic number

$$(1.1) \quad I(\psi) := \int_E c_1(VTE) c^n,$$

where  $VTE$  is the vertical tangent bundle of  $E$  and  $c \in H^2(E, \mathbb{R})$  is the coupling class of the fibration  $E \rightarrow S^2$  [3, 6].  $I(\psi)$  depends only on the homotopy class of

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the loop  $\psi$ . Moreover, the map

$$(1.2) \quad I : \psi \in \pi_1(\text{Ham}(N, \Omega)) \mapsto I(\psi) \in \mathbb{R}$$

is an  $\mathbb{R}$ -valued group homomorphism [4].

Our purpose is to study this characteristic number when  $N$  is a toric manifold and  $\psi$  is a 1-parameter subgroup of  $\text{Ham}(N)$  defined by the toric action. The referred 1-parameter subgroup is determined by an element  $\mathbf{b}$  in the integer lattice of the Lie algebra of the corresponding torus. On the other hand, a toric symplectic manifold is determined by its moment polytope. For a general polytope, a mass linear function on it is a linear function “whose value on the center of mass of the polytope depends linearly on the positions of the supporting hyperplanes” [7]. In this article, we will relate the vanishing of the number  $I(\psi)$  with the fact that  $\mathbf{b}$  defines a mass linear function on the polytope associated with the toric manifold. In the following paragraphs, we provide a more detailed exposition of this relation.

Let  $T$  be the torus  $(U(1))^n$  and  $\Delta = \Delta(\mathbf{n}, k)$  the polytope in  $\mathfrak{t}^*$  with  $m$  facets defined by

$$(1.3) \quad \Delta(\mathbf{n}, k) = \bigcap_{j=1}^m \{x \in \mathfrak{t}^* : \langle x, \mathbf{n}_j \rangle \leq k_j\},$$

where  $k_j \in \mathbb{R}$  and the  $\mathbf{n}_j \in \mathfrak{t}$  are the outward conormals to the facets. The facet defined by the equation  $\langle x, \mathbf{n}_j \rangle = k_j$  will be denoted  $F_j$ , and we put  $\text{Cm}(\Delta)$  for the mass center of the polytope  $\Delta$ .

In [7] the chamber  $\mathcal{C}_\Delta$  of  $\Delta$  is defined as the set of  $k' \in \mathbb{R}^m$  such that the polytope  $\Delta' := \Delta(\mathbf{n}, k')$  is analogous to  $\Delta$ ; that is, the intersection  $\bigcap_{j \in J} F_j$  is nonempty iff  $\bigcap_{j \in J} F'_j \neq \emptyset$  for any  $J \subset \{1, \dots, m\}$ . When we consider only polytopes which belong to the chamber of a fixed polytope we delete the  $\mathbf{n}$  in the notation introduced in (1.3).

Further, McDuff and Tolman introduced the concept of mass linear pair: Given the polytope  $\Delta$  and  $\mathbf{b} \in \mathfrak{t}$ , the pair  $(\Delta, \mathbf{b})$  is mass linear if the map

$$(1.4) \quad k \in \mathcal{C}_\Delta \mapsto \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle \in \mathbb{R}$$

is linear. That is,

$$(1.5) \quad \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j + C,$$

where  $R_j$  and  $C$  are constant.

Let us assume that  $\Delta$  is a Delzant polytope [1]. We shall denote by  $(M_\Delta, \omega_\Delta, \mu_\Delta)$  the toric manifold determined by  $\Delta$  ( $\mu_\Delta : M \rightarrow \mathfrak{t}^*$  being the corresponding moment map). Given  $\mathbf{b}$ , an element in the integer lattice of  $\mathfrak{t}$ , we shall write  $\psi_{\mathbf{b}}$  for the loop of Hamiltonian diffeomorphisms of  $(M_\Delta, \omega_\Delta)$  defined by  $\mathbf{b}$  through the toric action. We will let  $I(\Delta; \mathbf{b})$  for the characteristic number  $I(\psi_{\mathbf{b}})$ . When we consider only polytopes in the chamber of a given polytope, we will write  $I(k; \mathbf{b})$  instead of  $I(\Delta(k); \mathbf{b})$  for  $k$  in this chamber.

The group  $G$  of the translations defined by the elements of  $\mathfrak{t}^*$  acts freely on  $\mathcal{C}_\Delta$ . We put  $r := m - n$  for the dimension of the quotient  $\mathcal{C}_\Delta/G$ . Thus,  $r$  is the number of effective parameters which characterize the polytopes in  $\mathcal{C}_\Delta$  considered as “physical bodies”.

We will prove the following theorem:

**Theorem 1.** *Let  $(\Delta, \mathbf{b})$  be a pair consisting of a Delzant polytope in  $\mathfrak{t}^*$  and an element in the integer lattice of  $\mathfrak{t}$ . If  $r \leq 2$ , the following statements are equivalent*

- (a)  $I(k; \mathbf{b}) = 0$ , for all  $k \in \mathcal{C}_\Delta$ .
- (b)  $(\Delta, \mathbf{b})$  is a mass linear pair as in (1.5), with  $\sum_j R_j = 0$ .

In [12], by direct computation, we proved the equivalence between the vanishing of  $I(k; \mathbf{b})$  on  $\mathcal{C}_\Delta$  and the fact that  $(\Delta, \mathbf{b})$  is a mass linear pair, when  $\Delta$  satisfies any of the following conditions:

- (i) it is the trapezium associated with a Hirzebruch surface,
- (ii) it is a  $\Delta_p$  bundle over  $\Delta_1$  [7],
- (iii)  $\Delta$  is the truncated simplex associated with the one point blow up of  $\mathbb{C}P^n$ .

On the other hand, when  $\Delta$  is any of these polytopes (i)-(iii), the number  $r$  is equal to 2; thus, from Theorem 1 and the result of [12], it follows that condition  $\sum_j R_j = 0$  is satisfied by all the mass linear pairs  $(\Delta, \mathbf{b})$ . This fact can also be proved by direct calculation (Propositions 14, 18 and 21). So, Theorem 1, together with these Propositions, generalize the result proved in [12].

Although the homotopy type of the Hamiltonian groups  $\text{Ham}(N, \Omega)$  is known only for some symplectic manifolds [5], the invariant  $I$  allows us to identify non-trivial elements in  $\pi_1(\text{Ham}(N, \Omega))$ . As  $I$  is a *group homomorphism*, from Theorem 1, we deduce that a sufficient condition for  $\psi_{\mathbf{b}}$  to generate an infinite cyclic subgroup in  $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$  is that the above condition (b) does not hold for  $(\Delta, \mathbf{b})$ . More precisely, we have the following consequence of Theorem 1:

**Theorem 2.** *Given the Delzant polytope  $\Delta$  and  $\mathbf{b}$  an element in the integer lattice of  $\mathfrak{t}$ . If  $r \leq 2$  and  $(\Delta, \mathbf{b})$  is not mass linear, then  $\psi_{\mathbf{b}}$  generates an infinite cyclic subgroup in  $\pi_1(\text{Ham}(M_{\Delta(k)}, \omega_{\Delta(k)}))$ , for all  $k \in \mathcal{C}_\Delta$ .*

In the proof of Theorem 1, a formula for the characteristic number  $I(\psi_{\mathbf{b}})$  obtained in [11] plays a crucial role. This formula gives  $I(\psi_{\mathbf{b}})$  in terms of the integrals, on the facets of the polytope, of the normalized Hamiltonian function corresponding to the loop  $\psi_{\mathbf{b}}$  (see (2.9)). From this expression for  $I(\psi_{\mathbf{b}})$ , we will deduce a relation between the directional derivative of map (1.4) along the vector  $(1, \dots, 1)$  of  $\mathbb{R}^m$ , the Euclidean volume of  $\Delta(k)$  and  $I(k; \mathbf{b})$  (see (3.1)). From this relation, it is easy to complete the proof of Theorem 1.

This article is organized as follows: In Section 2, we study the characteristic number  $I(k; \mathbf{b})$ , when  $(\Delta, \mathbf{b})$  is a linear pair and  $k$  varies in the chamber of  $\Delta$ ; we prove that  $I(k; \mathbf{b})$  is a homogeneous polynomial of the  $k_j$  (Proposition 6).

In Section 3, we prove Theorem 1. In Proposition 11, a sufficient geometric condition for the Delzant polytope  $\Delta$  to admit a mass linear pair  $(\Delta, \mathbf{b})$  is given. For a Delzant polytope  $\Delta$ , Proposition 12 gives a necessary condition for the vanishing of  $I(k; \mathbf{b})$  on  $\mathcal{C}_\Delta$ . We also express  $\sum_j R_j$  in terms of the displacement of the center of mass  $\text{Cm}(\Delta(k))$  produced by the change  $k_j \rightarrow k_j + 1$  (Proposition 13).

Section 4 concerns the form which Theorem 2 adopts, when  $\Delta$  is a Delzant polytope of the particular types (i)-(iii) mentioned above (see Corollary 15, Theorems 17 and 20). We also prove that, in these particular cases, if  $(\Delta, \mathbf{b})$  is a mass linear pair, then  $\sum_j R_j = 0$ .

## 2. A CHARACTERISTIC NUMBER

Let us suppose that the polytope  $\Delta$  defined in (1.3) is a Delzant polytope in  $\mathfrak{t}^*$ . Following [2], we recall some points of the construction of  $(M_\Delta, \omega_\Delta, \mu_\Delta)$  from the

polytope  $\Delta$ . We put  $\tilde{T} := (S^1)^{m-n}$ . The  $\mathbf{n}_i$  determine weights  $w_j \in \tilde{\mathfrak{t}}^*$ ,  $j = 1, \dots, m$  for a  $\tilde{T}$ -action on  $\mathbb{C}^m$ . Then moment map for this action is

$$J : z \in \mathbb{C}^m \mapsto J(z) = \pi \sum_{j=1}^m |z_j|^2 w_j \in \tilde{\mathfrak{t}}^*.$$

The  $k_i$  define a regular value  $\sigma$  for  $J$ , and the manifold  $M_\Delta$  is the following orbit space

$$(2.1) \quad M_\Delta = \{z \in \mathbb{C}^m : \pi \sum_{j=1}^m |z_j|^2 w_j = \sigma\} / \tilde{T},$$

where the relation defined by  $\tilde{T}$  is

$$(2.2) \quad (z_j) \simeq (z'_j) \text{ iff there is } \mathbf{y} \in \tilde{\mathfrak{t}} \text{ such that } z'_j = z_j e^{2\pi i \langle w_j, \mathbf{y} \rangle} \text{ for } j = 1, \dots, m.$$

Identifying  $\tilde{\mathfrak{t}}^*$  with  $\mathbb{R}^r$ ,  $\sigma = (\sigma_1, \dots, \sigma_r)$  and each  $\sigma_a$  is a linear combination of the  $k_j$ .

Given a facet  $F$  of  $\Delta$ , we choose a vertex  $p$  of  $F$ . After a possible change in numeration of the facets, we can assume that  $F_1, \dots, F_n$  intersect at  $p$ . In this numeration  $F = F_j$ , for some  $j \in \{1, \dots, n\}$ .

If we write  $z_a = \rho_a e^{i\theta_a}$ , then the symplectic form can be written on  $\{[z] \in M : z_a \neq 0, \forall a\}$

$$(2.3) \quad \omega_\Delta = (1/2) \sum_{i=1}^n d\rho_i^2 \wedge d\varphi_i,$$

with  $\varphi_i$  an angular variable, linear combination of the  $\theta_a$ .

The action of  $T = (S^1)^n$  on  $M_\Delta$

$$(\alpha_1, \dots, \alpha_n)[z_1, \dots, z_m] := [\alpha_1 z_1, \dots, \alpha_n z_n, z_{n+1}, \dots, z_m]$$

endows  $M_\Delta$  with a structure of toric manifold. Identifying  $\mathfrak{t}^*$  with  $\mathbb{R}^n$ , the moment map  $\mu_\Delta : M_\Delta \rightarrow \mathfrak{t} = \mathbb{R}^n$  is defined by

$$(2.4) \quad \mu_\Delta([z]) = \pi(\rho_1^2, \dots, \rho_n^2) + (d_1, \dots, d_n),$$

where the constants  $d_i$  are linear combinations of the  $k_j$  and

$$(2.5) \quad \text{im } \mu_\Delta = \Delta.$$

The facet  $F = F_j$  of  $\Delta$  is the image by  $\mu_\Delta$  of the submanifold

$$D_j = \{[z_1, \dots, z_m] \in M_\Delta \mid z_j = 0\}.$$

We write  $x_i := \pi\rho_i^2$ , then

$$(2.6) \quad \int_{M_\Delta} (\omega_\Delta)^n = n! \int_\Delta dx_1 \dots dx_n.$$

Let  $\mathbf{b}$  be an element in the integer lattice of  $\mathfrak{t}$ . The normalized Hamiltonian of the circle action generated by  $\mathbf{b}$  is the function  $f$  determined by,

$$f = \langle \mu_\Delta, \mathbf{b} \rangle + \text{constant} \quad \text{and} \quad \int_{M_\Delta} f (\omega_\Delta)^n = 0.$$

That is,  $f = \langle \mu_\Delta, \mathbf{b} \rangle - \langle \text{Cm}(\Delta), \mathbf{b} \rangle$ , where

$$(2.7) \quad \langle \text{Cm}(\Delta), \mathbf{b} \rangle = \frac{\int_M \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n}{\int_M (\omega_\Delta)^n}.$$

Moreover,

$$(2.8) \quad \int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n = n! \int_\Delta \sum_{i=1}^n b_i x_i dx_1 \dots dx_n.$$

An expression for the value of the invariant  $I(\psi_{\mathbf{b}})$  in terms of integrals of the Hamiltonian function has been obtained in Section 4 of [11] (see also [10] and [9])

$$(2.9) \quad I(\Delta; \mathbf{b}) := I(\psi_{\mathbf{b}}) = -n \sum_{F \text{ facet}} N(F),$$

where the contribution  $N(F)$  of the above facet  $F = F_j$  (with  $j = 1, \dots, n$ ) is

$$(2.10) \quad \begin{aligned} N_j &:= N(F) = (n-1)! \int_{F_j} f dx_1 \dots d\hat{x}_j \dots dx_n \\ &= (n-1)! \left( \int_{F_j} \langle \mu_\Delta, \mathbf{b} \rangle dx_1 \dots d\hat{x}_j \dots dx_n - \langle \text{Cm}(\Delta), \mathbf{b} \rangle \int_{F_j} dx_1 \dots d\hat{x}_j \dots dx_n \right), \end{aligned}$$

with  $dx_1 \dots d\hat{x}_j \dots dx_n := dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n$ .

Given  $\Delta = \Delta(\mathbf{n}, k)$ , we consider the polytope  $\Delta' = \Delta(\mathbf{n}, k')$  obtained from  $\Delta$  by the translation defined by a vector  $a$  of  $\mathfrak{t}^*$ . As we said, we write  $I(k; \mathbf{b})$  and  $I(k'; \mathbf{b})$  for the corresponding characteristic numbers. According to the construction of the respective toric manifolds,

$$M_{\Delta'} = M_\Delta, \quad \omega_{\Delta'} = \omega_\Delta, \quad \mu_{\Delta'} = \mu_\Delta + a.$$

But the *normalized* Hamiltonians  $f$  and  $f'$  corresponding to the action of  $\mathbf{b}$  on  $M_\Delta$  and  $M_{\Delta'}$  are equal. Thus, it follows from (2.9) that  $I(k; \mathbf{b}) = I(k'; \mathbf{b})$ . More precisely, we have the evident proposition:

**Proposition 3.** *If  $a$  is an arbitrary vector of  $\mathfrak{t}^*$ , then  $I(k; \mathbf{b}) = I(k'; \mathbf{b})$ , for  $k'_j = k_j + \langle a, \mathbf{n}_j \rangle$ ,  $j = 1, \dots, m$ .*

By Proposition 3, we can assume that all  $d_j$  in (2.4) are zero for the determination of  $I(k; \mathbf{b})$ .

The following lemma is elementary:

**Lemma 4.** *If*

$$S_n(\tau) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq \tau, \quad 0 \leq x_j, \quad \forall j \right\},$$

then

$$\int_{S_n(\tau)} f(x_1, \dots, x_n) dx_1 \dots dx_n = \begin{cases} \frac{\tau^n}{n!}, & \text{if } f = 1 \\ c \frac{\tau^{n+c}}{(n+c)!}, & \text{if } f = x_i^c, \quad c = 1, 2 \\ \frac{\tau^{n+2}}{(n+2)!}, & \text{if } f = x_i x_j, \quad i \neq j. \end{cases}$$

More general, if  $c_1, \dots, c_n \in \mathbb{R}_{>0}$ , we put

$$S_n(c, \tau) := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n c_i x_i \leq \tau, \quad 0 \leq x_j, \quad \forall j \right\},$$

then

$$(2.11) \quad \int_{S_n(c, \tau)} dx_1 \dots dx_n = \frac{1}{n!} \prod_{i=1}^n \frac{\tau}{c_i}, \quad \int_{S_n(c, \tau)} x_j dx_1 \dots dx_n = \frac{1}{(n+1)!} \frac{\tau}{c_j} \prod_{i=1}^n \frac{\tau}{c_i}$$

Thus, in the particular case that  $\Delta = S_n(c, \tau)$ , the integral  $\int_{M_\Delta} (\omega_\Delta)^n$  is a monomial of degree  $n$  in  $\tau$ , and  $\int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n$  is a monomial of degree  $n+1$ .

We return to the general case in which  $\Delta$  is the polytope defined in (1.3). Its vertices are the solutions to

$$(2.12) \quad \langle x, \mathbf{n}_{j_a} \rangle = k_{j_a}, \quad a = 1, \dots, n;$$

hence, the coordinates of any vertex of  $\Delta$  are linear combinations of the  $k_j$ .

A hyperplane in  $\mathbb{R}^n$  through a vertex  $(x_1^0, \dots, x_n^0)$  of  $\Delta$  is given by an equation of the form

$$(2.13) \quad \langle x, \mathbf{n} \rangle = \langle x^0, \mathbf{n} \rangle =: \kappa.$$

Thus, the independent term  $\kappa$  is a linear combination (l. c.) of the  $k_j$ . Moreover, the coordinates of the common point of  $n$  hyperplanes

$$(2.14) \quad \langle x, \tilde{\mathbf{n}}_i \rangle = \kappa_i,$$

with  $\kappa_i$  l. c. of the  $k_j$  are also l. c. of the  $k_j$ .

By drawing hyperplanes through vertices of  $\Delta$  (or more generally, through points which are the intersection of  $n$  hyperplanes as (2.14)), we can obtain a family  $\{\beta S\}$  of subsets of  $\Delta$  such that:

a) Each  $\beta S$  is the transformed of a simplex  $S_n(b, \tau)$  by an element of the group of Euclidean motions in  $\mathbb{R}^n$ .

b) For  $\alpha \neq \beta$ ,  $\alpha S \cap \beta S$  is a subset of the border of  $\alpha S$ .

c)  $\bigcup_{\beta} \beta S = \Delta$ .

Thus, by construction, each facet of  $\beta S$  is contained in a hyperplane  $\pi$  of the form  $\langle x, \mathbf{n} \rangle = \kappa$ , with  $\kappa$  l. c. of the  $k_j$ .

On the other hand, the hyperplane  $\pi$  is transformed by an element of  $\text{SO}(n)$  in an hyperplane  $\langle x, \mathbf{n}' \rangle = \kappa$ . If  $\mathcal{T}$  is a translation in  $\mathbb{R}^n$  which applies  $S_n(b, \tau)$  onto  $\beta S$ , then this transformation maps  $(0, \dots, 0)$  in a vertex  $a = (a_1, \dots, a_n)$  of  $\beta S$ . So, the translation  $\mathcal{T}$  transforms  $\pi$  in  $\langle x, \mathbf{n} \rangle = \kappa + \langle a, \mathbf{n} \rangle =: \kappa'$ . As each  $a_j$  is a l. c. of the  $k_j$ , so is  $\kappa'$ . Hence, any element of the group of Euclidean motions in  $\mathbb{R}^n$  which maps  $S_n(b, \tau)$  onto  $\beta S$  transforms the hyperplane  $\pi$

$$(2.15) \quad \langle x, \mathbf{n}' \rangle = \kappa',$$

with  $\kappa'$  a l. c. of the  $k_j$ .

Let assume that  $(R\mathcal{T}_a)(S(b, \tau)) = \beta S$ , with  $R \in \text{SO}(n)$  and  $\mathcal{T}_a$  the translation defined by  $a$ . Then the oblique facet of  $S(b, \tau)$ , contained in the hyperplane  $\sum b_i x_i = \tau$ , is the image by  $T_{-a}R^{-1}$  of a facet of  $\beta S$ , which in turn is contained in a hyperplane of equation (2.15) ( $\kappa'$  being a l. c. of the  $k_j$ ). The argument of the preceding paragraph applied to  $R^{-1}$  and  $\mathcal{T}_{-a}$  proves that  $\tau$  is a l. c. of the  $k_j$ . Hence, by (2.11) the integral

$$\int_{\beta S} dx_1 \dots dx_n = \int_{S_n(b, \tau)} dx_1 \dots dx_n$$

is a monomial of degree  $n$  of a l. c. of the  $k_j$ . Thus,

$$(2.16) \quad \int_M (\omega_\Delta)^n = \sum_\beta \int_{\beta S} dx_1 \dots dx_n,$$

is a homogeneous polynomial of degree  $n$  of the  $k_j$ .

Similarly,

$$(2.17) \quad \int_{M_\Delta} \langle \mu_\Delta, \mathbf{b} \rangle (\omega_\Delta)^n$$

is a homogeneous polynomial of degree  $n + 1$  of the  $k_j$ . Analogous results hold for

$$\int_{F_j} dx_1 \dots d\hat{x}_j \dots dx_n, \quad \text{and} \quad \int_{F_j} \langle \mu_\Delta, \mathbf{b} \rangle dx_1 \dots d\hat{x}_j \dots dx_n.$$

From formulas (2.6)-(2.10) together with the preceding argument, it follows the following proposition:

**Proposition 5.** *Given a Delzant polytope  $\Delta$ , if  $\mathbf{b}$  belongs to the integer lattice of  $\mathfrak{t}$ , then  $I(k; \mathbf{b})$  is a rational function of the  $k_j$ , for  $k \in \mathcal{C}_\Delta$ .*

Analogously, we have

**Proposition 6.** *If  $(\Delta, \mathbf{b})$  is mass linear pair, then  $I(k; \mathbf{b})$  is a homogeneous polynomial in the  $k_j$  of degree  $n$ , when  $k \in \mathcal{C}_\Delta$ .*

We will use the following simple lemma in the proof of Theorem 1.

**Lemma 7.** *If  $\hat{k}_j = sk_j$  for  $j = 1, \dots, m$ , with  $s \in \mathbb{R}$ , then  $\text{Cm}(\Delta(\mathbf{n}, \hat{k})) = s \text{Cm}(\Delta(\mathbf{n}, k))$ .*

*Proof.* The vertices of  $\Delta(\mathbf{n}, k)$  are the solutions of (2.12) and the vertices of  $\Delta(\mathbf{n}, \hat{k})$  are the solutions of  $\langle x, \mathbf{n}_{j_a} \rangle = sk_{j_a}$ , with  $a = 1, \dots, n$ . Thus, the vertices of  $\Delta(\mathbf{n}, \hat{k})$  are those of  $\Delta(\mathbf{n}, k)$  multiplied by  $s$ .  $\square$

The Lemma also follows from the fact that (2.16) and (2.17) are homogeneous polynomials of degree  $n$  and  $n + 1$ , respectively.

### 3. PROOF OF THEOREM 1

Let us assume that the polytope  $\Delta$  defined by (1.3) is Delzant and let  $k$  be an element of  $\mathcal{C}_\Delta$ . We denote by  $M_{(k)}$ ,  $\omega_{(k)}$  and  $\mu_{(k)}$ , the manifold, the symplectic structure and the moment map (resp.) determined by  $\Delta(k)$ . The facets of  $\Delta(k)$  will be denoted by  $F_{(k)j}$ .

Let  $\mathbf{b}$  be an element in the integer lattice of  $\mathfrak{t}$ . We put

$$A_{(k)} := \int_{M_{(k)}} \langle \mu_{(k)}, \mathbf{b} \rangle (\omega_{(k)})^n, \quad B_{(k)} := \int_{M_{(k)}} (\omega_{(k)})^n.$$

By (2.6),  $\frac{1}{n!} B_{(k)}$  is the Euclidean volume of the polytope  $\Delta(k)$ . Given a facet  $F_{(k)j}$ , we can assume that  $j \in \{1, \dots, n\}$  (see third paragraph of Section 2). So,  $F_{(k)j}$  is defined by the equation  $x_j = 0$ . If we make an infinitesimal variation of the facet  $F_{(k)j}$ , by means of the translation defined by  $k_j \rightarrow k_j + \epsilon$  (keeping unchanged the other  $k_i$ ), then the volume of  $\Delta(k)$  changes according to

$$\frac{1}{n!} B_{(k)} \longrightarrow \frac{1}{n!} B_{(k)} + \epsilon \int_{F_{(k)j}} dx_1 \dots d\hat{x}_j \dots dx_n + O(\epsilon^2).$$

We write  $dX^j$  for  $dx_1 \dots \hat{dx}_j \dots dx_n$ . Thus,

$$\frac{\partial B_{(k)}}{\partial k_j} = n! \int_{F_{(k)j}} dX^j, \quad \frac{\partial A_{(k)}}{\partial k_j} = n! \int_{F_{(k)j}} \langle \mu_{(k)}, \mathbf{b} \rangle dX^j.$$

So, by (2.7),

$$\frac{\partial}{\partial k_j} \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{n!}{(B_{(k)})^2} \left( B_{(k)} \int_{F_{(k)j}} \langle \mu_{(k)}, \mathbf{b} \rangle dX^j - A_{(k)} \int_{F_{(k)j}} dX^j \right).$$

From (2.9) and (2.10), it follows

$$(3.1) \quad \sum_{j=1}^m \frac{\partial}{\partial k_j} \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{-1}{B_{(k)}} I(k; \mathbf{b}).$$

Thus, we have proved the following proposition:

**Proposition 8.**  $I(k; \mathbf{b}) = 0$  for all  $k \in \mathcal{C}_\Delta$  iff  $\sum_{j=1}^m \frac{\partial}{\partial k_j} \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = 0$ , for all  $k \in \mathcal{C}_\Delta$ .

Next, we will parametrize the quotient  $\mathcal{C}_\Delta/G$  (of classes of polytopes in  $\mathcal{C}_\Delta$  module translation) defined in the Introduction.

After a possible renumbering, we may assume that the intersection of facets  $F_1, \dots, F_n$  is a vertex of  $\Delta$ . Thus, the conormals  $\mathbf{n}_1, \dots, \mathbf{n}_n$  are linearly independent in  $\mathfrak{t}$ . So, given  $k \in \mathcal{C}_\Delta$ , there is a unique  $v \in \mathfrak{t}^*$ , such that,

$$(3.2) \quad \langle v, \mathbf{n}_i \rangle + k_i = 0, \quad i = 1, \dots, n.$$

(Expressing the  $\mathbf{n}_i$  in terms of a basis of  $\mathfrak{t}$  and  $v$  in the dual basis, (3.2) is a compatible and determined system of linear equations for the coordinates of  $v$ .) Moreover  $v = v(k)$  depends *linearly* of the  $k_i$ ; that is,  $\langle v(k), \mathbf{c} \rangle$  is a linear function of  $k_1, \dots, k_n$ , for all  $\mathbf{c} \in \mathfrak{t}$ .

If  $m - n = 2$ , we write

$$\lambda = k_{n+1} + \langle v(k), \mathbf{n}_{n+1} \rangle, \quad \tau = k_m + \langle v(k), \mathbf{n}_m \rangle,$$

where  $v(k)$  the element in  $\mathfrak{t}^*$  defined by (3.2). From the linearity of  $v$  with respect to the  $k_i$ , it follows that  $\lambda$  and  $\tau$  are *linear combinations* of  $k_1, \dots, k_m$ .

The polytope in  $\mathcal{C}_\Delta$  defined by  $(k'_1 = 0, \dots, k'_n = 0, \lambda, \tau)$  will be denoted by  $\Delta_0(\lambda, \tau)$ . It is the result of the translation of  $\Delta(k)$  by the vector  $v(k)$ ; i. e.,

$$(3.3) \quad \Delta_0(\lambda, \tau) = \Delta(k) + v(k).$$

Let  $\mathbf{b}$  an element in the integer lattice of  $\mathfrak{t}$ , we define the function  $g$  by

$$g(\lambda, \tau) := \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle.$$

The function  $g$  is defined on the pairs  $(\lambda, \tau)$  such that  $(0, \dots, 0, \lambda, \tau) \in \mathcal{C}_\Delta$ . By Lemma 7, it follows

$$g(s\lambda, s\tau) = sg(\lambda, \tau),$$

for any real number  $s$  such that  $(s\lambda, s\tau)$  belongs to the domain of  $g$ . This property implies that

$$(3.4) \quad g = \lambda \frac{\partial g}{\partial \lambda} + \tau \frac{\partial g}{\partial \tau}.$$

**Theorem 9.** If  $I(k; \mathbf{b}) = 0$ , for all  $k \in \mathcal{C}_\Delta$  and  $r = 2$ , then  $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j$ , with  $R_j$  constant (that is,  $(\Delta, \mathbf{b})$  is a mass linear pair) and  $\sum_j R_j = 0$ .



*Proof.* We set  $f(k_1, \dots, k_m) := \langle \text{Cm}(\Delta(k), \mathbf{b}) \rangle$ . It follows from (3.3) that

$$(3.5) \quad f(k) = g(\lambda, \tau) - \langle v(k), \mathbf{b} \rangle.$$

By the hypothesis and Proposition 8,

$$(3.6) \quad \sum_{j=1}^m \frac{\partial f}{\partial k_j} = 0.$$

Since

$$\sum_{j=1}^m \frac{\partial f}{\partial k_j} = \frac{\partial g}{\partial \lambda} \sum_{j=1}^m \frac{\partial \lambda}{\partial k_j} + \frac{\partial g}{\partial \tau} \sum_{j=1}^m \frac{\partial \tau}{\partial k_j} - \left\langle \frac{\partial v}{\partial k_j}, \mathbf{b} \right\rangle,$$

from (3.6) we deduce

$$(3.7) \quad p \frac{\partial g}{\partial \lambda} + q \frac{\partial g}{\partial \tau} - t = 0,$$

where  $p, q, t$  stand for the following constants

$$p = \sum_{j=1}^m \frac{\partial \lambda}{\partial k_j}, \quad q = \sum_{j=1}^m \frac{\partial \tau}{\partial k_j}, \quad t = \left\langle \frac{\partial v}{\partial k_j}, \mathbf{b} \right\rangle.$$

Since  $q\lambda - p\tau$  and  $t\tau - qg$  are first integrals of (3.7), the general solution of this equation is

$$(3.8) \quad g(\lambda, \tau) = (t/q)\tau + \Phi(q\lambda - p\tau),$$

where  $\Phi$  is a derivable function of one variable.

It follows from (3.4) and (3.8) that

$$(3.9) \quad \Phi(u) = u\Phi'(u).$$

Thus,  $\Phi(u) = \alpha u$ , with  $\alpha$  constant. We have for  $f$

$$f(k) = (b/q)\tau + \alpha(q\lambda - p\tau) - \langle v(k), \mathbf{b} \rangle.$$

In other words,  $f$  is a linear function of the  $k_j$ ; i.e.,  $f(k) = \sum_j R_j k_j$ , with  $R_j$  constant. From (3.6), it follows  $\sum_j R_j = 0$ .  $\square$

*Remark.* The proof of Theorem 9 can be adapted to the simpler case when  $r = 1$ . In this case, the function  $g(\lambda) = \langle \text{Cm}(\Delta_0(\lambda), \mathbf{b}) \rangle$ , satisfies  $p \frac{dg}{d\lambda} - t = 0$  and  $g(s\lambda) = sg(\lambda)$ . So,  $g(\lambda) = (t/p)\lambda$  and  $f(k) = (t/p)\lambda + \langle v(k), \mathbf{b} \rangle$  is a linear map of the variables  $k_j$ .

On the other hand, the proof of this theorem does not admit an adaptation to the case  $r > 2$ . In fact, the corresponding function  $\Phi$  would be a function of  $r - 1$  variables  $\Phi(u_1, \dots, u_{r-1})$ . The equation which corresponds to (3.9) in this case would be

$$\Phi = \sum_{i=1}^{r-1} u_i \frac{\partial \Phi}{\partial u_i}.$$

But this condition does not implies the linearity of  $\Phi$ .

When  $(\Delta, \mathbf{b})$  is a mass linear pair as in (1.5), by (3.1)

$$(3.10) \quad I(k; \mathbf{b}) = -B_{(k)} \sum_j R_j,$$

for all  $k \in \mathcal{C}_\Delta$ . From (3.10), we deduce the following proposition:

**Proposition 10.** *Let  $(\Delta, \mathbf{b})$  be a mass linear pair.  $I(k; \mathbf{b}) = 0$  for all  $k \in \mathcal{C}_\Delta$  iff  $\sum_j R_j = 0$ .*

**Proof of Theorem 1.** It is a direct consequence of Proposition 10, Theorem 9 and the Remark above.  $\square$

We will deduce a sufficient condition for a Delzant polytope  $\Delta$  to admit mass linear functions. We write

$$\dot{\text{Cm}}(\Delta(k)) := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(k + \check{\epsilon})),$$

with  $\check{\epsilon} = (\epsilon, \dots, \epsilon)$ .

**Proposition 11.** *If all points  $\dot{\text{Cm}}(\Delta(k))$ , for  $k \in \mathcal{C}_\Delta$ , belong to a hyperplane of  $(\mathbb{R}^n)^*$  with a conormal vector in  $\mathbb{Z}^n$  and  $r \leq 2$ , then  $\Delta$  admits a mass linear function.*

*Proof.* Let  $\mathbf{b} \in \mathbb{Z}^n$  be a conormal vector to the hyperplane, then

$$0 = \langle \dot{\text{Cm}}(\Delta(k)), \mathbf{b} \rangle = \left\langle \sum_j \frac{\partial}{\partial k_j} \text{Cm}(\Delta(k)), \mathbf{b} \right\rangle.$$

By (3.1),  $I(k, \mathbf{b}) = 0$ ; Theorem 9 applies and  $(\Delta, \mathbf{b})$  is a mass linear pair.  $\square$

**Proposition 12.** *Let  $\Delta$  be a Delzant polytope, such that  $k = 0$  belongs to the closure of  $\mathcal{C}_\Delta$ . If  $r \leq 2$ , a necessary condition for the vanishing of  $I(k; \mathbf{b})$  on  $\mathcal{C}_\Delta$  is*

$$\left\langle \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(\check{\epsilon})), \mathbf{b} \right\rangle = 0.$$

*Proof.* If  $I(k; \mathbf{b})$  vanishes on  $\mathcal{C}_\Delta$ , then  $(\Delta, \mathbf{b})$  is a linear pair, by Theorem 1. Thus,  $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_j R_j k_j + C$ , on  $\mathcal{C}_\Delta$ . So, given  $k \in \mathcal{C}_\Delta$  and  $\epsilon$  small enough

$$\langle \text{Cm}(\Delta(k + \check{\epsilon})), \mathbf{b} \rangle = \sum_j R_j k_j + \epsilon \sum_j R_j + C.$$

By Theorem 1,  $\sum_j R_j = 0$ . Thus, for any  $k \in \mathcal{C}_\Delta$ ,

$$\left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \text{Cm}(\Delta(k + \check{\epsilon})), \mathbf{b} \rangle = 0.$$

Taking the limit as  $k \rightarrow 0$ ,

$$0 = \lim_{k \rightarrow 0} \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \langle \text{Cm}(\Delta(k + \check{\epsilon})), \mathbf{b} \rangle = \left\langle \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{Cm}(\Delta(\check{\epsilon})), \mathbf{b} \right\rangle.$$

$\square$

Next, we will describe a geometric interpretation of the number  $\sum_j R_j$ . Given an arbitrary Delzant polytope  $\Delta$ . If  $a$  is a vector of  $\mathfrak{t}^*$ , then

$$(3.11) \quad \text{Cm}(\Delta(k')) = \text{Cm}(\Delta(k)) + a,$$

if  $k'_j = k_j + \langle a, \mathbf{n}_j \rangle$ .

We will denote by  $d$  the element of  $\mathfrak{t}^*$  defined by the following relation

$$(3.12) \quad \text{Cm}(\Delta(\tilde{k})) = \text{Cm}(\Delta(k)) + d,$$

with  $\tilde{k}_j = k_j + 1$  for all  $j$ .

From (3.11) and (3.12), we have

$$\text{Cm}(\Delta(k_j + \langle d, \mathbf{n}_j \rangle)) = \text{Cm}(\Delta(k_j)) + d = \text{Cm}(\Delta(\tilde{k}_j = k_j + 1)).$$

Now, we assume that  $(\Delta, \mathbf{b})$  is a mass linear pair. From (1.5), it follows

$$\langle \text{Cm}(\Delta(k_j + \langle d, \mathbf{n}_j \rangle)), \mathbf{b} \rangle = \sum R_j k_j + \sum R_j \langle d, \mathbf{n}_j \rangle + C.$$

$$\langle \text{Cm}(\Delta(k_j)) + d, \mathbf{b} \rangle = \sum R_j k_j + \langle d, \mathbf{b} \rangle + C, \quad \langle \text{Cm}(\Delta(\tilde{k}_j)), \mathbf{b} \rangle = \sum R_j k_j + \sum R_j + C.$$

These formulas allow us to state the following proposition, that gives an interpretation of the sum  $\sum_j R_j$  in terms of the variation of  $\text{Cm}(\Delta(k))$  with the  $k_j$ .

**Proposition 13.** *Let  $(\Delta, \mathbf{b})$  be a mass linear pair as in (1.5). Then,*

$$\sum_j R_j \langle d, \mathbf{n}_j \rangle = \langle d, \mathbf{b} \rangle = \sum_j R_j,$$

*d being the element of  $\mathfrak{t}^*$  defined by (3.12).*

#### 4. EXAMPLES

In this Section, we will deduce the particular form which adopts Theorem 2, when  $\Delta$  is a polytope of the types (i)-(iii) mentioned in the Introduction. For each case, we will determine the center of mass of the corresponding polytope  $\Delta(k)$  and the condition for  $(\Delta, \mathbf{b})$  to be a mass linear pair. We will dedicate a subsection to each type.

**4.1. Hirzebruch surfaces.** Given  $r \in \mathbb{Z}_{>0}$  and  $\tau, \lambda \in \mathbb{R}_{>0}$  with  $\sigma := \tau - r\lambda > 0$ , in [12] we considered the Hirzebruch surface  $N$  determined by these numbers.  $N$  is the quotient

$$\{z \in \mathbb{C}^4 : |z_1|^2 + r|z_2|^2 + |z_4|^2 = \tau/\pi, |z_2|^2 + |z_3|^2 = \lambda/\pi\} / \mathbb{T},$$

where the equivalence defined by  $\mathbb{T} = (S^1)^2$  is given by

$$(a, b) \cdot (z_1, z_2, z_3, z_4) = (az_1, a^r bz_2, bz_3, az_4),$$

for  $(a, b) \in (S^1)^2$ .

The manifold  $N$  equipped with the following  $(U(1))^2$  action

$$(\epsilon_1, \epsilon_2)[z_j] = [\epsilon_1 z_1, \epsilon_2 z_2, z_3, z_4],$$

is a toric manifold. The corresponding moment polytope  $\Delta$  is the trapezium in  $\mathbb{R}^2$  with vertices

$$(4.1) \quad P_1 = (0, 0), \quad P_2 = (0, \lambda), \quad P_3 = (\tau, 0), \quad P_4 = (\sigma, \lambda).$$

That is,  $N$  is the toric manifold  $M_\Delta$  determined by the trapezium  $\Delta$ .

As the conormals to the facets of  $\Delta$  are the vectors  $\mathbf{n}_1 = (-1, 0)$ ,  $\mathbf{n}_2 = (0, -1)$ ,  $\mathbf{n}_3 = (0, 1)$  and  $\mathbf{n}_4 = (1, r)$ , the facets of a generic polytope  $\Delta(k)$  in  $\mathcal{C}_\Delta$  are on the straights

$$-x = k_1, \quad -y = k_2, \quad y = k_3, \quad x + ry = k_4.$$

The vertices of  $\Delta(k)$  are the points

$$(-k_1, -k_2), \quad (-k_1, k_3), \quad (k_4 - rk_3, k_3), \quad (k_4 + rk_2, -k_2).$$

Thus, the translation in the plane  $x, y$  defined by  $(-k_1, -k_2)$  transforms the trapezium determined by the vertices (4.1) in  $\Delta(k)$ , if

$$(4.2) \quad \tau = k_4 + rk_2 + k_1, \quad \lambda = k_3 + k_2.$$

So,

$$(4.3) \quad \text{Cm}(\Delta(k)) = \text{Cm}(\Delta) + (-k_1, -k_2).$$

Moreover, the mass center of  $\Delta$  is

$$(4.4) \quad \text{Cm}(\Delta) = \left( \frac{3\tau^2 - 3r\tau\lambda + r^2\lambda^2}{3(2\tau - r\lambda)}, \frac{3\lambda\tau - 2r\lambda^2}{3(2\tau - r\lambda)} \right).$$

The chamber  $\mathcal{C}_\Delta$  consists of the points  $(k_1, \dots, k_4)$  such that  $\tau - r\lambda > 0$ , with  $\tau$  and  $\lambda$  given by (4.2). So, the point  $k = 0$  belongs to the closure of  $\mathcal{C}_\Delta$ . From (4.3), together with (4.2) and (4.4), it follows

$$(4.5) \quad \text{Cm}(\Delta(\epsilon)) = \left( \frac{r^2\epsilon}{12}, \frac{-r\epsilon}{6} \right),$$

where  $\epsilon = (\epsilon, \epsilon, \epsilon, \epsilon)$ . By Proposition 12, if  $I(k; \mathbf{b})$  with  $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$  vanishes on the chamber  $\mathcal{C}_\Delta$ , then  $rb_1 - 2b_2 = 0$ .

On the other hand, from (4.4) and (4.3), it follows

$$(4.6) \quad \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2}{3(2\tau - r\lambda)} - k_1b_1 - k_2b_2.$$

By (4.2), expression (4.6) is linear in the  $k_i$  iff

$$\frac{(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2}{3(2\tau - r\lambda)}$$

is linear in  $\tau, \lambda$ . That is, iff there exist constants  $A, B$  such that for all  $\tau, \lambda$

$$(3\tau^2 - 3r\tau\lambda + r^2\lambda^2)b_1 + (3\lambda\tau - 2r\lambda^2)b_2 = 3(2\tau - r\lambda)(A\tau + B\lambda).$$

From this relation, it follows the above condition  $rb_1 = 2b_2$ . In this case (4.6) reduces to

$$(4.7) \quad \langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{-b_1}{2}k_1 + \frac{b_1}{2}k_4.$$

Comparing (1.5) with (4.7), we obtain,  $R_1 = -R_4 = \frac{-b_1}{2}$ ,  $R_2 = R_3 = 0$ ; so,  $\sum_j R_j = 0$ . That is, the condition  $\sum_j R_j = 0$  holds for all the mass pairs  $(\Delta, \mathbf{b})$  when  $\Delta$  is the polytope associated to a Hirzebruch surface. Hence,

**Proposition 14.**  *$(\Delta, \mathbf{b})$  is a mass linear pair iff  $rb_1 = 2b_2$ . Moreover, in this case  $\sum_j R_j = 0$ .*

By Theorem 2, we have

**Corollary 15.** *If  $rb_1 \neq 2b_2$ , then  $\psi_{\mathbf{b}}$  generates an infinite cyclic subgroup in  $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$ .*

*Remark.*

We denote by  $\phi_t$  the following isotopy of  $M_\Delta$

$$\phi_t[z] = [e^{2\pi it}z_1, z_2, z_3, z_4].$$

$\phi$  is a loop in the Hamiltonian group of  $M_\Delta$ . By  $\phi'$  we denote the Hamiltonian loop

$$\phi'_t[z] = [z_1, e^{2\pi it}z_2, z_3, z_4].$$

In Theorem 8 of [10] we proved that  $I(\phi') = (-2/r)I(\phi)$ . If  $\mathbf{b} = (b_1, b_2) \in \mathbb{Z}^2$ , then

$$I(\psi_{\mathbf{b}}) = b_1I(\phi) + b_2I(\phi') = (b_1 - (2/r)b_2)I(\phi).$$

That is,  $I(\psi_{\mathbf{b}}) = 0$  iff  $rb_1 = 2b_2$ , which is in agreement with Proposition 14 and Theorem 1.

**4.2.  $\Delta_p$  bundle over  $\Delta_1$ .** Given the integer  $p > 1$ , as McDuff and Tolman in [7], we consider the following vectors in  $\mathbb{R}^{p+1}$

$$(4.8) \quad \mathbf{n}_i = -e_i, \ i = 1, \dots, p, \quad \mathbf{n}_{p+1} = \sum_{i=1}^p e_i, \quad \mathbf{n}_{p+2} = -e_{p+1}, \quad \mathbf{n}_{p+3} = e_{p+1} - \sum_{i=1}^p a_i e_i,$$

where  $e_1, \dots, e_{p+1}$  is the standard basis of  $\mathbb{R}^{p+1}$  and  $a_i \in \mathbb{Z}$ . We write

$$\mathbf{a} := (a_1, \dots, a_p) \in \mathbb{Z}^p, \quad A := \sum_{i=1}^p a_i, \quad \mathbf{a} \cdot \mathbf{a} = \sum_{i=1}^p a_i^2.$$

Let  $\lambda, \tau$  be real positive numbers with  $\lambda + a_i > 0$ , for  $i = 1, \dots, p$ . In this subsection we will consider the polytope  $\Delta$  in  $(\mathbb{R}^{p+1})^*$  defined by the above conormals  $\mathbf{n}_j$  and the following  $k_j$

$$(4.9) \quad k_1 = \dots = k_p = k_{p+2} = 0, \quad k_{p+1} = \tau, \quad k_{p+3} = \lambda.$$

This polytope will be also denote by  $\Delta_0(\lambda, \tau)$ . It is a  $\Delta_p$  bundle on  $\Delta_1$  (see [7]). When  $p = 2$ ,  $\Delta = \Delta_0(\lambda, \tau)$  is the prism whose base is the triangle of vertices  $(0, 0, 0)$ ,  $(\tau, 0, 0)$  and  $(0, \tau, 0)$  and whose ceiling is the triangle determined by  $(0, 0, \lambda)$ ,  $(\tau, 0, \lambda + a_1\tau)$  and  $(0, \tau, \lambda + a_2\tau)$ .

We assume that the above polytope  $\Delta$  is a Delzant polytope. The manifold (2.1) is in this case

$$M_\Delta = \{z \in \mathbb{C}^{p+3} : \sum_{i=1}^{p+1} |z_i|^2 = \tau/\pi, \quad -\sum_{j=1}^p a_j |z_j|^2 + |z_{p+2}|^2 + |z_{p+3}|^2 = \lambda/\pi\} / \simeq,$$

where  $(z_j) \simeq (z'_j)$  iff there are  $\alpha, \beta \in U(1)$  such that

$$z'_j = \alpha \beta^{-a_j} z_j, \ j = 1, \dots, p; \quad z'_{p+1} = \alpha z_{p+1}; \quad z'_k = \beta z_k, \ k = p+2, p+3.$$

Thus,  $M_\Delta$  is the total space of the fibre bundle  $\mathbb{P}(L_1 \oplus \dots \oplus L_p \oplus \mathbb{C}) \rightarrow \mathbb{C}P^1$ , where  $L_j$  is the holomorphic line bundle over  $\mathbb{C}P^1$  with Chern number  $a_j$ .

The symplectic form (2.3) is

$$\omega_\Delta = (1/2)(\sigma_1 + \dots + \sigma_p + \sigma_{p+2}),$$

where  $\sigma_k = d\rho_k^2 \wedge d\varphi_k$ .

And the moment map

$$(4.10) \quad \mu_\Delta([z]) = (x_1, \dots, x_p, x_{p+2}),$$

where  $x_i := \pi \rho_i^2$ .

**Proposition 16.** *The coordinates  $\bar{x}_j$  of  $\text{Cm}(\Delta_0(\lambda, \tau))$  are given by:*

$$(4.11) \quad \bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}, \quad \text{for } k = 1, \dots, p.$$

$$(4.12) \quad \bar{x}_{p+2} = \frac{1}{2} \frac{(p+1)(p+2)\lambda^2 + 2(p+2)A\lambda\tau + (\mathbf{a} \cdot \mathbf{a} + A^2)\tau^2}{(p+2)((p+1)\lambda + A\tau)}.$$

*Proof.* Since the points  $[z] \in M_\Delta$  satisfy  $|z_{p+2}|^2 \leq \lambda/\pi + \sum_{j=1}^p a_j |z_j|^2$ , by (2.6) and Lemma 4 we have

$$(4.13) \quad \int_{M_\Delta} (\omega_\Delta)^{p+1} = (p+1)! \int_{S_p(\tau)} \left( \lambda + \sum_{j=1}^p a_j x_j \right) = (p+1)! \left( \frac{\lambda \tau^p}{p!} + \frac{\tau^{p+1} A}{(p+1)!} \right).$$

Similarly, for  $k = 1, \dots, p$

$$(4.14) \quad \int_{M_\Delta} x_k (\omega_\Delta)^{p+1} = (p+1)! \left( \frac{\lambda \tau^{p+1}}{(p+1)!} + \frac{\tau^{p+2}}{(p+2)!} \sum_{j \neq k} a_j + \frac{2\tau^{p+2} a_k}{(p+2)!} \right).$$

The  $k$ -th coordinate of  $\text{Cm}(\Delta)$ ,  $\bar{x}_k$ , is the quotient of (4.14) by (4.13); that is,

$$\bar{x}_k = \frac{\tau}{p+2} \frac{\lambda(p+2) + \tau(A + a_k)}{\lambda(p+1) + \tau A}.$$

For the  $p+2$ -coordinate of  $\text{Cm}(\Delta)$ , we need to calculate  $\int_M x_{p+2} (\omega_\Delta)^{p+1}$ . By Lemma 4

$$(4.15) \quad \frac{1}{(p+1)!} \int_M x_{p+2} (\omega_\Delta)^{p+1} = \frac{1}{2} \int_{S_p(\tau)} \left( \lambda + \sum_{j=1}^p a_j x_j \right)^2 \\ = \frac{1}{2} \left( \frac{\lambda^2 \tau^p}{p!} + \frac{2A\lambda \tau^{p+1}}{(p+1)!} + \frac{(\mathbf{a} \cdot \mathbf{a} + A^2) \tau^{p+2}}{(p+2)!} \right).$$

Formula (4.12) is a consequence of (4.13) together with (4.15).  $\square$

The translation in  $(\mathbb{R}^{p+1})^*$  defined by the vector  $(-k_1, \dots, -k_p, -k_{p+2})$  transforms the hyperplanes  $\langle x, \mathbf{n}_{p+3} \rangle = \lambda$  and  $\langle x, \mathbf{n}_{p+1} \rangle = \tau$  in

$$(4.16) \quad \langle x, \mathbf{n}_{p+3} \rangle = \lambda - k_{p+2} + \sum_{j=1}^p a_j k_j, \quad \langle x, \mathbf{n}_{p+1} \rangle = \tau - \sum_{j=1}^p k_j,$$

respectively.

Let  $\Delta(k)$  be a polytope with  $k = (k_1, \dots, k_{p+3})$  generic in the chamber  $\mathcal{C}_\Delta$ . From (4.16), it follows that  $\Delta(k)$  is the image of the polytope  $\Delta_0(\lambda, \tau)$  by the translation determined by  $(-k_1, \dots, -k_p, -k_{p+2})$ , whenever

$$(4.17) \quad k_{p+2} - \sum_{j=1}^p a_j k_j + k_{p+3} = \lambda, \quad \sum_{j=1}^p k_j + k_{p+1} = \tau.$$

In this case,

$$(4.18) \quad \text{Cm}(\Delta(k)) = \text{Cm}(\Delta_0(\lambda, \tau)) - (k_1, \dots, k_p, k_{p+2}).$$

According to (4.17), the coordinates of the mass center  $\text{Cm}(\Delta(\check{\epsilon}))$ , with  $\check{\epsilon} = (\epsilon, \dots, \epsilon)$ , can be obtained substituting in (4.11) and in (4.12)  $\lambda$  by

$$\epsilon - \sum_{j=1}^p a_j \epsilon + \epsilon = (2 - A)\epsilon$$

and  $\tau$  by  $(p+1)\epsilon$ , and finally take into account (4.18). These operations give

$$\bar{x}_j(\Delta(\check{\epsilon})) = \frac{\epsilon}{2(p+2)} ((p+1)a_j - A), \quad j = 1, \dots, p \\ \bar{x}_{p+2}(\Delta(\check{\epsilon})) = \frac{\epsilon}{4(p+2)} (-A^2 + (p+1)(\mathbf{a} \cdot \mathbf{a})).$$

Given  $\mathbf{b} = (b_1, \dots, b_p, b) \equiv (\hat{\mathbf{b}}, \dot{\mathbf{b}})$ , with  $\hat{\mathbf{b}} = (b_1, \dots, b_p, 0)$  and  $\dot{\mathbf{b}} = (0, \dots, 0, b)$ .

$$\left\langle \frac{d}{d\epsilon} \Big|_{\epsilon=0} \text{Cm}(\Delta(\epsilon)), \mathbf{b} \right\rangle = \frac{1}{4(p+2)} \left( (p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - b\mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \right),$$

where  $\mathbf{a} \cdot \hat{\mathbf{b}} = \sum_{j=1}^p a_j b_j$  and  $B = \sum_{j=1}^p b_j$ .

By Proposition 12, we have:

**Theorem 17.** *Let  $\Delta$  be the  $\Delta_p$  bundle over  $\Delta_1$  defined by (4.8) and (4.9). Given  $\mathbf{b} = (\hat{\mathbf{b}}, \dot{\mathbf{b}}) \in \mathbb{Z}^{p+1}$ , if*

$$(p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - b\mathbf{a} \cdot \mathbf{a}) - A(2B + bA) \neq 0,$$

*then  $\psi_{\mathbf{b}}$  defines an infinite cyclic subgroup in the fundamental group  $\pi_1(\text{Ham}(M_{\Delta}, \omega_{\Delta}))$ .*

It is straightforward to check that

$$(4.19) \quad (p+1)(2\mathbf{a} \cdot \hat{\mathbf{b}} - b\mathbf{a} \cdot \mathbf{a}) - A(2B + bA) = 0$$

is also a sufficient condition for  $(\Delta, \mathbf{b})$  to be a mass linear pair.

Since

$$\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \hat{\mathbf{b}} \rangle + \langle \text{Cm}(\Delta_0(\lambda, \tau)), \dot{\mathbf{b}} \rangle,$$

if (4.19) holds, using (4.11) and (4.12), one obtains

$$\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = \frac{b\lambda}{2} + \left( \frac{b}{2} \frac{(\mathbf{a} \cdot \mathbf{a} + A^2)}{(p+2)A} + \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A} \right) \tau.$$

By (4.18), for  $k \in \mathcal{C}_{\Delta}$ ,

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle - \sum_{j=1}^p b_j k_j - b k_{p+2},$$

with  $\lambda$  and  $\tau$  given by (4.17).

If  $\mathbf{b} = \hat{\mathbf{b}}$ , the condition (4.19) reduces to  $(p+1)\mathbf{a} \cdot \hat{\mathbf{b}} = AB$  and

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A} \left( \sum_{j=1}^p k_j + k_{p+1} \right) - \sum_{j=1}^p b_j k_j.$$

Hence,  $\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum R_j k_j$ , where

$$R_j = \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A} - b_j, \quad j = 1, \dots, p; \quad R_{p+1} = \frac{(\mathbf{a} \cdot \hat{\mathbf{b}} + AB)}{(p+2)A}, \quad R_{p+2} = R_{p+3} = 0.$$

So,

$$\sum_{j=1}^{p+3} R_j = \frac{(p+1)\mathbf{a} \cdot \hat{\mathbf{b}} - AB}{(p+2)A} = 0.$$

A similar calculation for the case  $\mathbf{b} = \dot{\mathbf{b}}$  shows that the corresponding  $\sum_j R_j$  vanishes. That is,

**Proposition 18.** *Let  $\Delta$  be a  $\Delta_p$  bundle over  $\Delta_1$ . If  $(\Delta, \mathbf{b})$  is a mass linear pair, then  $\sum_j R_j = 0$ .*

For  $p = 2$ , let  $\mathbf{b}$  be the following linear combination of the conormal vectors  $\mathbf{b} = \gamma_1 \mathbf{n}_1 + \gamma_2 \mathbf{n}_2 + \gamma_3 \mathbf{n}_3$  with  $\gamma_1 + \gamma_2 + \gamma_3 = 0$ . By (4.8),  $\mathbf{b} = (b_1, b_2, 0)$  with  $b_1 = \gamma_3 - \gamma_1$ ,  $b_2 = \gamma_3 - \gamma_2$ . In this case condition (4.19) reduces to

$$3(a_1 b_1 + a_2 b_2) = (a_1 + a_2)(b_1 + b_2).$$

Or in terms of the  $\gamma_i$

$$(4.20) \quad a_1 \gamma_1 + a_2 \gamma_2 = 0.$$

This is a necessary and sufficient condition for  $(\Delta, \mathbf{b})$  to be mass linear. This result is the statement of Lemma 4.8 in [7].

**4.3. One point blow up of  $\mathbb{C}P^n$ .** In this subsection  $\Delta \equiv \Delta_0(\lambda, \tau)$  will be

$$(4.21) \quad \Delta = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq \tau, 0 \leq x_i, x_n \leq \lambda \right\},$$

where  $\tau, \lambda \in \mathbb{R}_{>0}$  and  $\sigma := \tau - \lambda > 0$ . That is,  $\Delta$  is the polytope obtained truncating the simplex  $S_n(\tau)$ , defined in Lemma 4, by a “horizontal” hyperplane through the point  $(0, \dots, 0, \lambda)$ . The manifold  $M_\Delta$  associated with  $\Delta$  is the one point blow up of  $\mathbb{C}P^n$ .

The mass center of the simplex  $S_n(\tau)$  is the point

$$(4.22) \quad \text{Cm}(S_n(\tau)) = \frac{\tau}{n+1} w,$$

with  $w = (1, \dots, 1)$ .

As the volume of  $S_n(\tau)$  is  $\tau^n/n!$ , it follows from (4.22)

$$(\tau^n - \sigma^n) \text{Cm}(\Delta) = \tau^n \frac{\tau}{n+1} w - \sigma^n \left( \frac{\sigma}{n+1} w + \lambda e_n \right).$$

That is,

$$(4.23) \quad \text{Cm}(\Delta) = \frac{1}{\tau^n - \sigma^n} \left( \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} \right) w - \lambda \sigma^n e_n \right).$$

Given  $k = (k_1, \dots, k_{n+2}) \in \mathcal{C}_\Delta$ , the facets of  $\Delta(k)$  are in the following hyperplanes:

$$(4.24) \quad -x_j = k_j, j = 1, \dots, n; \quad \sum_{i=1}^p x_k = k_{n+1}; \quad x_{n+1} = k_{n+2}.$$

As in the preceding subsections,

$$(4.25) \quad \Delta(k) = \Delta_0(\lambda, \tau) - (k_1, \dots, k_n),$$

provided  $\lambda = k_n + k_{n+2}$  and  $\tau = \sum_{i=1}^{n+1} k_i$ .

The pair  $(\Delta, \mathbf{b} = (b_1, \dots, b_n))$  is mass linear iff there exist  $A, B, C \in \mathbb{R}$  such that

$$\sum_{j=1}^{n-1} b_j \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} + b_n \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} - (\tau - \sigma) \sigma^n \right) = (A\tau + B\sigma + C)(\tau^n - \sigma^n),$$

for all  $\tau, \sigma$  “admissible”. A simple calculation proves the following proposition:

**Proposition 19.** *The pair  $(\Delta, \mathbf{b})$  is mass linear iff*

$$b_n = \frac{1}{n} \sum_{j=1}^{n-1} b_j.$$



From Theorem 2 together with Proposition 19, it follows the following theorem:

**Theorem 20.** *If  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$  and  $\sum_{j=1}^{n-1} b_j \neq nb_n$ , then  $\psi_{\mathbf{b}}$  generates an infinite cyclic subgroup in  $\pi_1(\text{Ham}(M_{\Delta}, \omega_{\Delta}))$ .*

For  $k \in \mathcal{C}_{\Delta}$ , by (4.25)

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle - \sum_{j=1}^n b_j k_j.$$

If  $(\Delta, \mathbf{b})$  is a mass linear pair, by (4.23) and Proposition 19, we have  $\langle \text{Cm}(\Delta_0(\lambda, \tau)), \mathbf{b} \rangle = b_n \tau$ . Thus,

$$\langle \text{Cm}(\Delta(k)), \mathbf{b} \rangle = \sum_{j=1}^{n+1} R_j k_j,$$

where  $R_j = b_n - b_j$ , for  $j = 1, \dots, n$  and  $R_{n+1} = b_n$ . Hence, we have the following proposition:

**Proposition 21.** *Let  $\Delta$  be the polytope obtained by truncating the standard  $n$ -simplex  $S_n(\tau)$  by a horizontal hyperplane. If  $(\Delta, \mathbf{b})$  is a mass linear pair, then  $\sum_j R_j = 0$ .*

*Remark.* When  $n = 3$  the toric manifold  $M_{\Delta}$  is

$$M_{\Delta} = \{z \in \mathbb{C}^5 : |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_5|^2 = \tau/\pi, |z_3|^2 + |z_4|^2 = \lambda/\pi\} / \mathbb{T},$$

where the action of  $\mathbb{T} = (U(1))^2$  is defined by

$$(4.26) \quad (a, b)(z_1, z_2, z_3, z_4, z_5) = (az_1, az_2, abz_3, bz_4, az_5),$$

for  $a, b \in U(1)$ .

We consider the following loops in the Hamiltonian group of  $(M_{\Delta}, \omega_{\Delta})$

$$\begin{aligned} \psi_t[z] &= [z_1 e^{2\pi i t}, z_2, z_3, z_4, z_5], \quad \psi'_t[z] = [z_1, z_2 e^{2\pi i t}, z_3, z_4, z_5], \\ \tilde{\psi}_t[z] &= [z_1, z_2, z_3 e^{2\pi i t}, z_4, z_5]. \end{aligned}$$

In [11] (Remark in Section 4), we gave formulas that relate the characteristic numbers associated with these loops

$$I(\psi) = I(\psi') = (-1/3)I(\tilde{\psi}).$$

So, for  $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{Z}^3$ ,

$$(4.27) \quad I(\psi_{\mathbf{b}}) = (b_1 + b_2 - 3b_3)I(\psi).$$

By Proposition 19, the vanishing of  $I(\psi_{\mathbf{b}})$  in (4.27) is equivalent to the fact that  $(\Delta, \mathbf{b})$  is a mass linear pair. This equivalence is a new checking of Theorem 1.

## REFERENCES

- [1] T. Delzant, *Hamiltoniens périodique et images convexes de l'application moment*. Bull. Soc. Math. France **116** (1988), 315-338.
- [2] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian  $T^n$ -spaces*. Birkhäuser, Boston, (1994).
- [3] V. Guillemin, L. Lerman, S. Sternberg, *Symplectic fibrations and multiplicity diagrams*. Cambridge U.P., Cambridge, (1996).
- [4] F. Lalonde, D. McDuff, L. Polterovich, *Topological rigidity of Hamiltonian loops and quantum homology*. Invent. Math. **135** (1999), 369-385.

- [5] D. McDuff, *A survey of the topological properties of symplectomorphism groups*. Topology, geometry and quantum field theory, 173-193, London Math. Soc. Lecture Note Ser., **308**. Cambridge U. P., Cambridge, (2004).
- [6] D. McDuff, D. Salamon, *Introduction to symplectic topology*, Clarendon Press, Oxford, (1998).
- [7] D. McDuff, S. Tolman, *Polytopes with mass linear functions. I*. Int. Math. Res. Not. IMRN **8** (2010) 1506-1574.
- [8] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms*, Birkhäuser, Basel, (2001).
- [9] E. Shelukhin, *Remarks on invariants of Hamiltonian loops*. J. Topol. Anal. **2** (2010) 277-325.
- [10] A. Viña, *A characteristic number of Hamiltonian bundles over  $S^2$* . J. Geom. Phys. **56** (2006), 2327-2343.
- [11] A. Viña, *Hamiltonian diffeomorphisms of toric manifolds and flag manifolds*. J. Geom. Phys. **57** (2007), 943-965.
- [12] A. Viña, *A characteristic number of bundles determined by mass linear pairs*. [arXiv:0809.1506 \[math.SG\]](#).

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